Rigidly Rotating Strings in Stationary Axisymmetric Spacetimes

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Abstract

In this paper we study a motion of a rigidly rotating Nambu-Goto test string in a stationary axisymmetric background spacetime. As special examples we consider rigid rotation of strings in a flat spacetime, where explicit analytic solutions can be obtained, and in the Kerr spacetime where we find an interesting new family of test string solutions. We present a detailed classification of these solutions in the Kerr background.

1 Introduction

Cosmic strings are cosmologically interesting objects [1] and the motion of strings in a curved background is a subject which has recently been intensively discussed [2, 3]. If one neglects the gravitational effects of a string and assumes that its thickness is zero, then the string configuration is a timelike minimal surface that is an extremum of the Nambu-Goto action. One of the interesting physical applications of the general theory is study of the interaction of cosmic strings with a black hole. If a (infinitely) long cosmic string passes nearby a black hole it can be captured [4]. Final stationary configurations of a trapped string were analyzed in [5] and their complete analytic description was obtained.

In the general case a stationary string in a stationary spacetime is defined as a timelike minimal surface that is tangent to the Killing vector generating time translations. In the Kerr-Newman metric the equations describing a stationary string allow separation of variables [6, 7, 8] and can be solved exactly [6]. In this paper we generalize these results to a wider class of string configurations. Namely we study rigidly rotating strings in a stationary axisymmetric background spacetime. A rigidly rotating string is a string which at different moments of time has the same form so that its configuration at later moment of time can be obtained by the rigid rotation of the initial configuration around the axis of symmetry. Denote by $\xi_{(t)}$ and $\xi_{(\phi)}$ Killing vectors that are generators of time translation and rotation. The timelike minimal worldsheets which represent a stationary rigidly rotating string are characterized by the property that the special linear combination $\xi_{(t)} + \Omega\xi_{(\phi)}$ is tangent to the worldsheet. Our aim is to study such configurations in a stationary axisymmetric spacetime.

The paper is organized as follows. General equations for a stationary rigidly rotating string in a stationary spacetime are obtained and analyzed in Section 2. As the simplest application we obtain explicit analytical solutions describing rotating strings in a flat spacetime (Section 3). One of the interesting results is the possibility of the rigid rotation of the string with (formally) superluminal velocity, i.e. when $r\Omega > 1$ (r is the distance from the axis of rotation). A simple explanation of this phenomenon is given in Section 3. Section 4 devoted to rigidly rotating strings in the Kerr spacetime. To conclude Section 4 we present a classification of this new family of test string solutions in the Kerr spacetime.

2 General Equations

Consider a stationary axisymmetric spacetime. Such a spacetime possesses at least two commuting Killing vectors: $\xi_{(t)}$ and $\xi_{(\phi)}$. If a spacetime is asymptotically flat the vector $\xi_{(t)}$ is singled out by the requirement that it is timelike at infinity. The vector $\xi_{(\phi)}$ is spacelike at infinity and it is singled out by the property that its integral curves are closed lines. The metric for a stationary axisymmetric spacetime can be written in the form

$$ds^2 = -V\left[dt - w d\phi\right]^2 + \frac{1}{V}\left[\rho^2 d\phi^2 + e^{2\gamma}(d\rho^2 + dz^2)\right], \tag{1}$$

where V, w and γ are functions of the coordinates ρ and z only. This is the so called the Papapetrou form of the metric for stationary axisymmetric spacetimes (see Ref. [9] for example). In these coordinates $\xi^\mu_{(t)} = \delta^\mu_t$ and $\xi^\mu_{(\phi)} = \delta^\mu_\phi$.

Denote by Σ a two-dimensional timelike minimal surface representing the motion of a string in this spacetime and denote by S_t the spatial slice $t = \text{const}$. The intersection of Σ with the surface S_t is a one-dimensional line γ_t representing the string configuration at the time t. We define a rigid cosmic string as one whose shape and extent (but not necessarily position) are independent of the coordinate time t. If x^i are spatial coordinates (for metric (1) (ρ, z, ϕ)) then γ_t is given by the equations $x^i = x^i(\sigma, t)$, where σ is a parameter along the string. Since $\xi_{(\phi)}$ is tangent to S_t it is a generator of symmetry transformations (spatial rotations) acting on S_t. It is evident that this transformation preserves the form and the shape of the string γ_t. Our assumption that the string at the moment t is obtained by a rigid rotation from the string γ_{t_0} can be written as

$$\rho(\sigma,t) = \rho(\sigma,t_0)\,, \quad z(\sigma,t) = z(\sigma,t_0)\,, \quad \phi(\sigma,t) = \phi(\sigma,t_0) + f(t,t_0)\,. \tag{2}$$

Moreover we assume uniform rotation, so that $f(t, t_0) = \Omega(t - t_0)$, where Ω is a constant angular velocity. It is evident that the following combination $\chi^\mu = \xi^\mu_{(t)} + \Omega\xi^\mu_{(\phi)}$ of the Killing vectors $\xi^\mu_{(t)}$ and $\xi^\mu_{(\phi)}$ is tangent to the worldsheet Σ of a uniformly rotating string.

In a region where χ^μ is timelike one can define a set of Killing observers whose four-velocities are $u^\mu = \chi^\mu / |\chi^2|^{1/2}$. This set of observers form a rigidly rotating reference frame that is the frame moving with a constant angular velocity Ω. One could choose to define a rigidly rotating string as a string which was fixed in form and position in the frame of some Killing observer

with angular velocity Ω. It can be shown that if all the string is located in the region where χ^μ is timelike this definition is equivalent to that given above. But, as we shall demonstrate later, situations are possible when a rigidly rotating string lies in a region where the Killing vector χ^μ is spacelike yet its world sheet surface Σ remains timelike. With this possibility in mind we will use the former definition of the rigid string rotation.

We begin by performing the following coordinate transformation:

$$\varphi = \phi - \Omega t, \tag{3}$$

where Ω is a constant. Metric (1) now takes the form

$$ds^2 = -V[(1-\Omega w)dt - w d\varphi]^2 + \frac{1}{V}\left[\rho^2(d\varphi + \Omega dt)^2 + e^{2\gamma}(d\rho^2 + dz^2)\right] \tag{4}$$

and the Killing vector χ has components $\chi^\mu = (1,0,0,0)$. The Killing trajectories of χ (that might be timelike or spacelike) are: $\rho, z, \varphi =$const .

A configuration of a test cosmic string in a given gravitational background is represented by a timelike two-surface Σ (a world sheet) that satisfies the Nambu-Goto equations of motion. A world sheet Σ can be described in the parametric form $x^\mu = x^\mu(\zeta^A)$, where x^μ are spacetime coordinates, and ζ^A $(A = 0,1)$ are the coordinates on the world sheet. For these coordinates we shall also use the standard notation $(\zeta^0, \zeta^1) = (\tau, \sigma)$. The Nambu-Goto action is given by

$$\mathcal{S}\left[x^\mu\right] = -\mu \int d^2\zeta \sqrt{-G}, \tag{5}$$

where G is the determinant of the induced metric on the world sheet $G_{AB} = g_{\mu\nu} x^\mu_{,A} x^\nu_{,B}$ and μ is the string tension.

For a stationary world sheet configuration one can choose parameters (τ, σ) in such a way that

$$x^\mu(\zeta^A) = (\tau + f(\sigma), \rho(\sigma), z(\sigma), \varphi(\sigma)), \tag{6}$$

where f is some function of σ. The determinant of the induced metric G_{AB} on the world sheet is

$$G = \frac{e^{2\gamma}\chi^2}{V}\left(\rho'^2 + z'^2\right) - \rho^2 \varphi'^{\,2}. \tag{7}$$

where

$$\chi^2 = -V + 2\Omega w V + \Omega^2(\rho^2/V - w^2 V), \tag{8}$$

Note that neither the function f nor its derivative f' appear in the action so they may be specified freely. It is convenient to choose f so that the induced metric is diagonal, i.e. $G_{\tau\sigma} = g_{\mu\nu} x^{\mu}_{,\tau} x^{\nu}_{,\sigma} = 0$. In this case we find f must be chosen to satisfy the condition

$$f' = -\frac{Vw + \Omega(\rho^2/V - w^2 V)}{\chi^2}\, \varphi'. \tag{9}$$

A stationary string configuration (6) provides an extremum for the reduced Nambu-Goto action

$$E = \mu \int d\sigma \sqrt{\frac{e^{2\gamma}(-\chi^2)}{V}\left(\rho'^2 + z'^2\right) + \rho^2 \varphi'^{\,2}}. \tag{10}$$

Hence a stationary string configuration $x^i = (\rho(\sigma), z(\sigma), \varphi(\sigma))$ is a geodesic line in a three-dimensional space with the metric

$$dh^2 = \frac{e^{2\gamma}(-\chi^2)}{V}\left(d\rho^2 + dz^2\right) + \rho^2 d\varphi^2\,. \tag{11}$$

The Nambu-Goto equations for a stationary rigidly rotating string are

$$\partial_\sigma\left(\frac{\chi^2 e^{2\gamma}}{\sqrt{-G}V}\rho'\right) = \frac{1}{\sqrt{-G}}\left[\frac{1}{2}\frac{\partial}{\partial\rho}\left(\frac{\chi^2 e^{2\gamma}}{V}\right)\left(\rho'^2 + z'^{\,2}\right) - \rho\varphi'^{\,2}\right], \tag{12}$$

$$\partial_\sigma\left(\frac{\rho^2\varphi'}{\sqrt{-G}}\right) = 0, \tag{13}$$

$$\partial_\sigma\left(\frac{\chi^2 e^{2\gamma}}{\sqrt{-G}V}z'\right) = \frac{1}{2\sqrt{-G}}\frac{\partial}{\partial z}\left(\frac{\chi^2 e^{2\gamma}}{V}\right)\left(\rho'^2 + z'^{\,2}\right), \tag{14}$$

where G is given by (7). Equation (13) can be integrated immediately to give

$$\varphi'^{\,2} = \frac{-\chi^2}{V}\left(\frac{L^2 e^{2\gamma}}{\rho^2(\rho^2 - L^2)}\right)(\rho'^2 + z'^{\,2}). \tag{15}$$

Here L is a constant of the integration. The constant L is associated with the φ-independence of the Lagrangian and is related to the angular momentum of the string. In what follows we choose L to be non-negative.

These equations are invariant under reparameterization $\sigma \to \tilde{\sigma} = \tilde{\sigma}(\sigma)$. In the region where $\rho' \neq 0$ one can use this ambiguity to put $\sigma = \rho$. For this choice equations (12)-(14) reduce to

$$\left(\frac{d\varphi}{d\rho}\right)^2 = \frac{-\chi^2 L^2 e^{2\gamma}}{\rho^2(\rho^2 - L^2)V}\left[1 + \left(\frac{dz}{d\rho}\right)^2\right], \tag{16}$$

$$\frac{d}{d\rho}\left(\frac{\chi^2 e^{2\gamma}}{\sqrt{-\tilde{G}V}}\frac{dz}{d\rho}\right) = \frac{1}{2\sqrt{-\tilde{G}}}\frac{\partial}{\partial z}\left(\frac{\chi^2 e^{2\gamma}}{V}\right)\left[1 + \left(\frac{dz}{d\rho}\right)^2\right], \tag{17}$$

where

$$-\tilde{G} = \frac{-G}{\rho'^{\,2}} = \frac{\rho^2\chi^2 e^{2\gamma}}{V(L^2 - \rho^2)}\left[1 + \left(\frac{dz}{d\rho}\right)^2\right]. \tag{18}$$

The solutions represent a timelike two-surface provided the determinant G is negative definite. Thus we see that the rigidly rotating strings are confined to regions (for $V > 0$) where

$$I \equiv \frac{L^2 - \rho^2}{\chi^2} > 0. \tag{19}$$

When $L^2 = \rho^2$ the world sheet has a turning point in ρ as a function of φ.

In general, in order to ensure rigid rotation of a string, an external force must act on it. For example, one could assume that a string has heavy monopoles at the end and that a magnetic field is applied to force them to move along a circle. In this case a solution of equations (16)-(17) describes the motion of the string interior. In order to escape a discussion of the details of the motion of the end points we shall use the maximal extensions of the string solutions, continuing them until they meet the surface where $\chi^2 = 0$. Since the invariant I changes its sign at this surface, the minimal surface describing the rigidly rotating string ceases to be timelike here. The end points of such a maximally extended string move with the velocity of light along this surface. In this paper, for brevity, we call such solutions “open” strings. In what follows we restrict ourselves to the study of such “open” strings and do not discuss how these solutions can arise when real forces are acting on the string or when a part of the string is involved in a rigid rotation.

Our assumption of rigidity implies that the coordinates ρ and z of the end points of the string remain fixed. Under these conditions the end points of an

"open" string are located on the surfaces where $\chi^2 = 0$. In a flat spacetime this timelike surface is a cylinder located at the radial distance Ω^{-1} from the axis of symmetry. In the general case we shall refer to the surfaces where $\chi^2 = 0$ as "null cylinders". Note that if $L^2 - \rho^2$ vanishes at the same point as χ^2 it is possible for the world sheet to pass through the surface $\chi^2 = 0$ and remain regular and timelike.

In order to find a rigidly rotating string configuration one needs to fix functions $V(\rho, z)$, $\gamma(\rho, z)$, and $w(\rho, z)$ that specify geometry. It is quite interesting (as was remarked by de Vega and Egusquiza [10]) that if the metric (1) allows a discrete symmetry $z \to -z$, then equations (16) and (17) always have a special solution, namely a string configuration described by the relations $z = 0$ and $\varphi = \text{const}$. De Vega and Egusquiza called these straight rigidly rotating strings in axially-symmetric stationary spacetimes "planetoid" solutions.

3 Rotating Strings in Flat Spacetime

Our main goal is a study of rigidly rotating strings in a spacetime of a rotating black hole. But before considering this problem we make a few remarks concerning rigidly rotating strings in a flat spacetime. We recover the Minkowski metric

$$ds^2 = -dt^2 + d\rho^2 + dz^2 + \rho^2 d\phi \tag{20}$$

from (1) by setting the metric functions $V = 1$ and $w = \gamma = 0$. We also have $\chi^2 = \Omega^2 \rho^2 - 1$. Since the metric is independent of z one can integrate (17) once to reduce the equations of motion to the form

$$\frac{d\varphi}{d\rho} = \pm \frac{L\ (1 - \Omega^2\,\rho^2)}{\rho\sqrt{(1 - p^2 + L^2\,\Omega^2)\ \rho^2 - \Omega^2\,\rho^4 - L^2}}, \tag{21}$$

$$\frac{dz}{d\rho} = \pm \frac{p\,\rho}{\sqrt{(1 - p^2 + L^2\,\Omega^2)\,\rho^2 - \Omega^2\,\rho^4 - L^2}}, \tag{22}$$

where p is a constant of integration. These equations can be solved analytically.

In order for (21) and (22) to be real-valued, ρ is constrained to lie in the interval $0 < \rho_- \le \rho < \rho_+$ where the upper and lower bounds are given by,

$$\rho_\pm = \Omega^{-1}\sqrt{(B \pm C)/2}\,, \tag{23}$$

where,

$$B = 1 - p^2 + L^2\,\Omega^2, \text{ and } C = \sqrt{B^2 - 4\,\Omega^2\,L^2}\,. \tag{24}$$

The equations for $z(\rho)$ and $\varphi(\rho)$ can be integrated readily (the substitution $u = \rho^2$ reduces these to standard integrals) with solutions,

$$\begin{aligned} z_\pm(\rho) &= \mp\frac{p}{2\Omega}\left(\arcsin\frac{B - 2\,\Omega^2\,\rho^2}{C} - \frac{\pi}{2}\right), \qquad (25)\\ \varphi_\pm(\rho) &= \pm\frac{1}{2}\left\{\arcsin\frac{B\,\rho^2 - 2\,L^2}{C\,\rho^2}\right.\\ &\qquad \left. + \,\Omega\,L\,\arcsin\frac{B - 2\,L^2\,\rho^2}{C} + \frac{\pi}{2}\,(1 - \Omega\,L)\right\} \qquad (26)\end{aligned}$$

where for convenience we have chosen the initial conditions $z_\pm(\rho_-) = \varphi_\pm(\rho_-) = 0$.

It is instructive to examine the special case where $p = 0$ further where the solutions are confined to the $z = \text{const}$ plane. The solution (26) can be rewritten in the form

$$\varphi_\pm(\rho) = \pm\left(\arctan k\,\xi - k^{-1}\arctan\xi\right)\,, \tag{27}$$

where $k = (\Omega L)^{-1}$ and $\xi = \Omega\sqrt{(\rho^2 - L^2)/(1 - \Omega^2\rho^2)}$.

In order for solutions to exist, the invariant $I \equiv (L^2 - \rho^2)/\chi^2$ must be non-negative. Thus there are a number of cases to resolve. We know that the string can end only on the null cylinder where χ^2 vanishes, i.e. $\rho = 1/\Omega$. We also see that the string may have a turning point at $\rho = L$. When $L = 0$ we recover the rigidly rotating straight strings of De Vega and Egusquiza [10]. When $L > 0$ there are two cases;

1. $L < 1/\Omega$: the string lies in the region $L < \rho < 1/\Omega$, has end-points at $\rho = 1/\Omega$ and a turning point at $\rho = L$ (see Figure 1),

2. $L > 1/\Omega$: the string lies in the region $L > \rho > 1/\Omega$. It has end-points at $\rho = 1/\Omega$ and a turning point at $\rho = L$ (see Figure 2).

(The case $L = 1/\Omega$ is excluded since $I < 0$ and hence no solution exists.)

In the latter case the Killing vector χ is spacelike. Nonetheless the world-sheet is timelike; in fact the tangent vector $x^\mu_{,\sigma}$ is timelike in this region.

However the solution lies in the region $L > \rho > 1/\Omega$ and appears, by comparing $t = \text{const}$ slices in non-rotating coordinates, to move at "superluminal velocities" (except at the end points which move at the speed of light). This is in apparent contradiction with the observation that the world sheet is timelike.

The puzzle is clarified if we note that the apparent velocity of the string in the surface $t = \text{const}$ is not the physical velocity of the string. Recall that the Nambu-Goto action is invariant under world-sheet reparameterizations. This reparameterization can be used to generate a "motion" of the string along itself, which evidently is physically irrelevant. In other words only velocity normal to the string world-sheet has physical meaning (see e.g. [1]).

Hence, on the $t = \text{const}$ hypersurfaces, we must to consider the component of the apparent string velocity *normal* to the string configuration. The normal component of the velocity is the physical component. The apparent three velocity, v^i $(i = 1, 2, 3)$, of the string in (ρ, z, ϕ) coordinates as measured by a static observer at infinity is

$$v^i = \Omega(0, 0, 1) \tag{28}$$

and has magnitude $v = \rho\Omega$.

Its projection on the normal to the string $u^i_\perp$ in the $(t = \text{const})$ plane is

$$u^i_\perp = \frac{\Omega}{1 + \rho^2 \varphi'^{\,2}}(-\rho^2\varphi', 0, 1), \tag{29}$$

and the magnitude of this normal velocity is

$$u^2_\perp = \frac{\rho^2\Omega^2}{1 + \rho^2\varphi'^{\,2}} = \frac{\Omega^2(L^2 - \rho^2)}{\Omega^2 L^2 - 1}. \tag{30}$$

Thus we see that if $1/\Omega^2 > \rho^2 > L^2$ (case 1) then $u^2_\perp < 1$ as expected. Furthermore if $L^2 > \rho^2 > 1/\Omega^2$ (the apparently "superluminal" case 2) we see that $u^2_\perp < 1$ also. The physical velocity of the string is subluminal in all cases where the solution exists.

This phenomenon is evidently of a quite general nature. In order to separate these two different types of rigid rotation of strings we will call the motion "superluminal" if they are tangent to χ with $\chi^2 > 0$ and "subluminal" if the world-sheet is tangent to χ with $\chi^2 < 0$.

4 Rigidly Rotating Strings in the Kerr Spacetime

4.1 Equations of Motion

In Boyer-Lindquist coordinates the Kerr metric is given by:

$$\begin{aligned} ds^2 = &- \frac{\Delta}{\Sigma}\left[dt - a\sin^2\theta d\phi\right]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \\ &+ \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\phi - adt\right]^2, \end{aligned} \tag{31}$$

where $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. This spacetime is stationary and axisymmetric with two Killing vectors: $\xi^{\mu}_{(t)} = \delta^{\mu}_{t}$ and $\xi^{\mu}_{(\phi)} = \delta^{\mu}_{\phi}$.

The relationship between the Boyer-Linquist coordinate functions and the Papapetrou coordinate functions is straightforward; the time coordinate t and the angular coordinate ϕ that appear in both the metric (1) and the metric (31) are simply identified and

$$\rho^2 = \Delta\sin^2\theta\,, \qquad z = (r-M)\cos\theta. \tag{32}$$

In terms of the Boyer-Lindquist coordinates the Papapetrou metric functions for the Kerr metric are

$$V = \frac{\Delta - a^2\sin^2\theta}{\Sigma}, \tag{33}$$

$$w = \frac{2Mra\sin^2\theta}{\Delta - a^2\sin^2\theta}, \tag{34}$$

$$e^{2\gamma} = \frac{\Delta - a^2\sin^2\theta}{\Delta\cos^2\theta + (r-M)^2\sin^2\theta}. \tag{35}$$

We can now write down the string equations of motion for the Kerr spacetime in the Boyer-Linquist coordinates. The string configuration is determined by two functions $\varphi(r)$ and $\theta(r)$ which satisfy

$$\left(\frac{d\varphi}{dr}\right)^2 = \frac{\tilde{G}L^2}{\Delta^2\sin^4\theta} \tag{36}$$

$$\frac{\Sigma\chi^2}{\sqrt{\tilde{G}}}\frac{d}{dr}\left(\frac{\Sigma\chi^2}{\sqrt{\tilde{G}}}\frac{d\theta}{dr}\right) = \frac{\cos\theta}{\Delta\sin^3\theta}Z\,, \tag{37}$$

where

$$Z = L^2 + b^2 \left(1 + \frac{\Delta(1 - a^2\Omega^2 \sin^2\theta)}{\Sigma\chi^2} \right) \tag{38}$$

$$b^2 = L^2 - \Delta \sin^2\theta \,, \qquad \tilde{G} = \frac{\Sigma\chi^2 \sin^2\theta}{b^2} \left[1 + \Delta \left(\frac{d\theta}{dr} \right)^2 \right] . \tag{39}$$

$$\chi^2 = \frac{\sin^2\theta \left[a^2 + 4\Omega M r a + \Omega^2((r^2 + a^2)^2 - \Delta a^2 \sin^2\theta) \right] - \Delta}{\Sigma} \tag{40}$$

We were not able to solve this system analytically. Moreover one can show (see Appendix) that the only case when the variables in the problem under consideration can be separated by the Hamilton-Jacobi method (at least in these coordinates) is when $\Omega = 0$, so that the problem reduces to the one considered in [6]. In what follows we consider the case when a rotating string is located in the equatorial plane, which allows more detailed analysis.

4.2 Rigidly Rotating Strings in the Equatorial Plane

For the motion of the string in the equatorial plane $\theta = \pi/2$ equation (37) is satisfied identically (both the left and right hand sides vanish) and equation (36) takes the form

$$\left(\frac{d\varphi}{dr} \right)^2 = \frac{\Sigma L^2 \chi^2}{\Delta^2 (L^2 - \Delta)} . \tag{41}$$

Here

$$\chi^2 = \frac{F}{r} \,, \qquad F = \Omega^2 r^3 + (a^2\Omega^2 - 1) r + 2M(a\Omega - 1)^2 . \tag{42}$$

Solutions exist only if the right-hand side of (41) is non-negative and hence (for $L \neq 0$)

$$I \equiv \frac{L^2 - \Delta}{\chi^2} \geq 0. \tag{43}$$

The positivity of the invariant I also guarantees that the world-sheet of the string is a regular timelike surface (cf. (19)).

To simplify the analysis of the structure of the null cylinder surfaces instead of r, a, L,Ω and F we introduce the dimensionless variables

$$\rho = \frac{r}{M}\,, \quad \alpha = \frac{a}{M}\,, \quad \lambda = \frac{L}{M}\,, \quad \omega = M\Omega\,, \quad \text{F=Mf}\,. \tag{44}$$

In these variables

$$\chi^2 = \frac{f}{\rho}\,, \quad f = \omega^2\rho^3 + (\alpha^2\omega^2 - 1)\rho + 2(\alpha\omega - 1)^2\,. \tag{45}$$

The surface where $L^2 - \Delta$ vanishes corresponds, in general, to turning points of the string in r. Now $L^2 - \Delta$ has only one zero outside the horizon, namely $r_0 = M\rho_0$ with

$$\rho_0 = 1 + \sqrt{1 + \lambda^2 - \alpha^2}\,. \tag{46}$$

For $r > r_0$, $L^2 - \Delta < 0$, and for $r < r_0$, $L^2 - \Delta > 0$.

Firstly we note that when $L = 0$ then we obtain the planetoid solution of De Vega and Egusquiza [10]. In this case the solution is a rigidly rotating straight string with end points on the null cylinders $r = r_1$ and $r = r_2$ where $r_2 > r_1$ are the zeroes of χ^2. Note that for a given Ω if these zeroes do not exist (so that χ is spacelike everywhere) then there are no such rigidly rotating *straight* strings.

In the general case the endpoints of an "open" string must be located on the null cylinders where $\chi^2 = 0$. Note one can think of the equation $f(\rho, \omega, \alpha) = 0$ as a quadratic in ω for fixed ρ and α. The zeroes of (45), that is the solutions of the equation $f(\rho, \omega, \alpha) = 0$, are

$$\omega_\pm = \frac{2\alpha \pm \rho\sqrt{\rho^2 - 2\rho + \alpha^2}}{\rho^3 + \alpha^2\rho + 2\alpha^2}. \tag{47}$$

They bound the interval $\omega_-(\rho) < \omega < \omega_+(\rho)$ where χ is timelike at a given radius $r = M\rho$. We note that inside the horizon where $\rho^2 - 2\rho + \alpha^2 < 0$, it is not possible for χ to be timelike or null except within or on the inner Cauchy horizon. Since we are interested in the motion of the strings in the black hole exterior from now on we restrict ourselves by solutions in the region $\rho > \rho_+$ where

$$\rho_+ = 1 + \sqrt{1 - \alpha^2}. \tag{48}$$

Equation (47) shows that $\omega_\pm|_{r_+} = \omega_{BH} \equiv \alpha/(\rho_+^2 + \alpha^2)$ (ω_{BH}/M is the angular velocity of the black hole). At large distances $\omega_\pm = \pm 1/\rho$ reproduces

flat space behavior. For fixed α the function $\omega_+(\rho,\alpha)$ has a maximum, and the function $\omega_-(\rho,\alpha)$ has a minimum. The points of the extrema can be defined as joint solutions of the equation $f(\rho,\omega,\alpha) = 0$ and the equation $\partial f/\partial\rho|_{(\omega,\alpha)} = 0$. The latter equation implies that

$$\rho = \pm\rho_m\, , \qquad \rho_m = \frac{\sqrt{1-\omega^2\alpha^2}}{\sqrt{3}|\omega|}\, . \tag{49}$$

A simultaneous solution of this relation and equation (47) defines a maximal value $\omega_{max}(\alpha)$ of ω_+ and a minimal value $\omega_{min}(\alpha)$ of ω_-. We conclude that for a given value of the rotation parameter α the Killing vector χ can be timelike only if $\omega_{min}(\alpha) < \omega < \omega_{max}(\alpha)$. If there exists a region where the Killing vector χ is timelike, this region is inside interval $\rho_1 < \rho < \rho_2$, and one has $\rho_{min}(\alpha) < \rho_1$ and $\rho_{max}(\alpha) > \rho_2$ (where $\rho_1(\omega,\alpha) < \rho_2(\omega,\alpha)$ are the zeroes of the polynomial f). Here $\rho_{min}(\alpha)$ and $\rho_{max}(\alpha)$ are given by (49) with $\omega = \omega_{min}(\alpha)$ and $\omega = \omega_{max}(\alpha)$, respectively.

We can arrive at the same conclusion by slightly different reasoning which will allow us to make further simplifications. For fixed α and ω the function $f(\rho,\omega,\alpha)$ is a cubic polynomial in ρ . It tends to $\pm\infty$ as $\rho \to \pm\infty$ and $f(0) \geq 0$ ($f(0) = 0$ only if $\omega\alpha = 1$). For $\alpha|\omega| \geq 1$ it is monotonic, and hence always positive at $\rho > 0$. For $\alpha|\omega| < 1$ the function $f(\rho)$ has a minimum at $\rho = \rho_m$ and a maximum at $\rho = -\rho_m$, where ρ_m is given by (49). At the minimum point, f takes the value

$$f_m = f(\rho_m,\omega) = 2(\omega\alpha-1)^2\left[1-\frac{1}{3\sqrt{3}|\omega|}\sqrt{\frac{(1+\omega\alpha)^3}{1-\omega\alpha}}\right]\, . \tag{50}$$

The minimum value f_m vanishes if the following equation is satisfied

$$\alpha(\alpha^2+27)\omega^3 + (3\alpha^2-27)\omega^2 + 3\alpha\omega + 1 = 0\, . \tag{51}$$

Solutions of this equations $\omega(\alpha)$ are also solutions of the two equations $f = 0$ and $\partial_r f = 0$, and hence they coincide with $\omega_{min}(\alpha)$ and $\omega_{max}(\alpha)$.

The numerical solution of equation (51) is shown in Figure 3. Line a represents solution ω_{max} and line b represents solution ω_{min}. These lines begin at $\alpha = 0$ at their Schwarzschild values $\pm 3^{-3/2}$ and reach values 1/2 and $-1/7$ respectively for the extremely rotating black hole. Figure 4 shows the corresponding radii ρ_{max} (curve a) and ρ_{min} (curve b) as the functions of the rotation parameter α.

The third branch c in Figure 3, which intersects a at $\alpha = 1$, corresponds to the minimum values of ω inside the Cauchy horizon. Line d is the solution of the equation $\omega\alpha = 1$. For the values of the parameters in the $(\alpha - \omega)$ plane lying in the region outside two shaded strips function f is positive for any $\rho > 0$. For the values of the parameters inside two shaded strips function f has two roots, $0 < \rho_1 < \rho_2$, and f is negative for ρ lying between the roots. The upper shaded region corresponds to the situation where the roots of f lie within the inner Cauchy horizon. The lower shaded region is the area where the two roots of f lie outside the event horizon.

Having obtained this information on the structure of the null cylinder surfaces we now discuss the different types of motion of a rigidly rotating string in the equatorial plane of the Kerr spacetime. The simplest situation clearly occurs when χ has no zeroes in the region $\rho \geq \rho_+$ (where ρ_+ is defined by equation (48)). This happens for values of the parameters which lie outside the shaded region restricted by lines a and b in the $(\alpha - \omega)$-plane (see Figure 3). In this case there is only one allowed type of solution $\rho_0 > \rho_+$ corresponding to solutions in the region $\rho_+ \leq \rho \leq \rho_0$. These configurations begin and end in the black hole and have a turning point at $\rho = \rho_0$ in its exterior (the form of these solutions is qualitatively similar to that of the solution shown in figure 6). Such a solution may describe a closed loop-like string, part of which has been swallowed by a black hole. Centrifugal forces connected with the rotation allow the other part of the string to remain in the black hole exterior.

For the values of the parameters lying inside the shaded region restricted by lines a and b in the $(\alpha - \omega)$-plane the situation is somewhat more complicated. In this case χ^2 has two zeroes that we denote by ρ_1 and ρ_2 $(\rho_+ < \rho_1 < \rho_2)$. The Killing vector χ is timelike in the region $\rho_1 < \rho < \rho_2$. Since the invariant I defined by (43) must be positive there are then a number of different possibilities depending upon the choice of the angular momentum parameter $\lambda \geq 0$.

1. $\rho_0 < \rho_1$. The invariant I is positive either if (a) $\rho_1 < \rho < \rho_2$ or (b) $\rho < \rho_0$. In the former case $f < 0$ and the motion of the string is "subluminal", with the ends of the string at ρ_1 and ρ_2 (Figure 5). In the latter case the motion of the string is "superluminal", the string begins and ends on the black hole and has a radial turning point ρ_0 in the black hole exterior (Figure 6).

2. $\rho_1 < \rho_0 < \rho_2$. The invariant I is positive either if (a) $\rho_0 < \rho < \rho_2$ or (b) $\rho < \rho_1$. In the former case $f < 0$ and the motion of the string is "subluminal", with both ends of the string at ρ_2 and ρ_0 being radial turning points (two examples (with $\lambda = 0.5$ and $\lambda = 2.0$) are shown in Figure 7). In the latter case the motion of the string is "superluminal" and string begins at the horizon ρ_+ and ends at ρ_1 (Figure 8).

3. $\rho_2 < \rho_0$. The invariant I is positive either if (a) $\rho_2 < \rho < \rho_0$ or (b) $\rho < \rho_1$. In both cases $f > 0$ and the motion is "superluminal". In the former case the ends of the string are at ρ_2 and a radial turning point is at ρ_0 (an example of such a configuration with $\lambda = 4.0$ is given in Figure 7). In the latter case the string configurations are similar to the one shown in Figure 8.

For "subluminal" motion the apparent velocity is less than the velocity of light, for "superluminal" motion the apparent velocity is greater than the velocity of light. In both cases the physical (orthogonal to the string) velocity is less than the velocity of light. We described the origin of this phenomenon in Section 3. We recall that in the above analysis we restricted ourselves to the case of rotating strings. In the absence of rotation stationary string configurations (both equatorial and off-equatorial) allow a complete description (see [6]).

Besides these main categories of motion there are possible different boundary cases when ρ_0 coincides either with ρ_1 or with ρ_2. These cases require special analysis. For special values of the parameters one might expect that a string passes through (and beyond) these points remaining regular and timelike (see Figure 9 for example). A similar situation was analysed in [5] for special configurations of strings which pass through the event horizon.

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Appendix

In this appendix we analyze separability of the Hamilton-Jacobi equations describing rigidly rotating strings in the Kerr spacetime. Since a rigidly rotating string in the axisymmetric stationary spacetime provides the minimum of the energy functional (10) and its configuration is a geodesic in a three-dimensional space with metric (11) one can use the following form of the Hamilton-Jacobi equation for defining such configurations

$$\frac{\partial S}{\partial \sigma} + \frac{1}{2} h_{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} = 0\,. \tag{52}$$

Here h^{ij} is the metric inverse to h_{ij} given by (11).

For the special case of Minkowski spacetime, where $V = 1$, $w = \gamma = 0$, (and setting $m = 1$), the Hamilton-Jacobi equations are trivially separable

$$S = -\frac{1}{2} m^2 \sigma + L\,\varphi + pz + A(\rho)\,, \tag{53}$$

and function $A(\rho)$ obeys the equation

$$\left(\frac{dA(\rho)}{d\rho}\right)^2 + p^2 + \left(\frac{L}{\rho^2} - 1\right) \left(1 - \rho^2\,\Omega^2\right) = 0\,. \tag{54}$$

By solving these equations and varying the action S with respect to separation constants and m one gets the complete set of equations that is equivalent to (21) and (22).

Turning now to the case of Kerr spacetime, it is easier to apply transformation (3) directly to the Kerr metric (31) and repeat the steps outlined above. It is straightforward, again, to show that the components of the spatial metric are,

$$h_{rr} = \Delta^{-1} h_{\theta\theta}\,, \quad h_{\theta\theta} = \Delta\, T^2 - \sin^2\theta\, R^2\,, \quad h_{\varphi\varphi} = \Delta \sin^2\theta \tag{55}$$

where,

$$T(\theta) = 1 - a\,\Omega \sin^2\theta\,, \quad R(r) = a - \Omega\,(r^2 + a^2)\,. \tag{56}$$

Assuming the separation of variables in this metric and working with an action of the form,

$$S = -\frac{1}{2} m^2 \sigma + L\,\varphi + A(r) + B(\theta) \tag{57}$$

the Hamilton-Jacobi equation yields

$$\begin{aligned}\Delta \quad & \left(\frac{dA(r)}{dr}\right)^2 + \left(\frac{dB(\theta)}{d\theta}\right)^2 + \frac{L^2\,T^2}{\sin^2\theta} \\ - \quad & \frac{L^2\,R^2}{\Delta} - m^2\,\Big[\Delta\,T^2 - R^2\sin^2\theta\Big] = 0. \qquad (58)\end{aligned}$$

The L^2 terms are effectively separated. Only the m^2 term requires consideration. In order to complete the separation of variables, it is required that,

$$m^2\,\Big[\Delta\,T^2 - R^2\sin^2\theta\Big] = K_1(r) + K_2(\theta). \qquad (59)$$

Expanding the bracket,

$$\begin{aligned}&\Big[\Delta\,T^2 - R^2\sin^2\theta\Big] = \Delta - a^2\sin^2\theta \qquad (60)\\ &+\Omega\sin^2\theta\left\{\Omega\,\left(\Delta\,a^2\sin^2\theta - \left(r^2+a^2\right)^2\right) + 2\,a\,\left(2\,M\,r - Q^2\right)\right\}.\end{aligned}$$

The first two terms are separated. Simple analysis shows that the last one cannot be. That is why in order to provide separability one must put $\Omega = 0$. The separation of variables for this case was studied earlier [6]. In the special case where the string is confined to the equatorial plane, the Hamilton-Jacobi equations are trivially separable since the function $B(\theta)$ is eliminated from the outset. It is then straightforward to obtain the results of section 4.2.

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Figure Captions

Figure 1. String configurations in flat spacetime for $p = 0$ with $L < 1/\Omega$. Solid lines represent strings for 4 different values $L = 0.05, 0.25, 0.5$, and 0.9 of the angular momentum. A dashed line is a null cylinder $\rho = 1/\Omega$ (here $\Omega = 1$). The arrow in this, and subsequent, figures indicates the direction of rotation of the strings.

Figure 2. String configurations in flat spacetime for $p = 0$ with $L > 1/\Omega$. Solid lines represent strings for 4 different values $L = 1.1, 1.5, 1.75$, and 1.95 of the angular momentum. A dashed line is a null cylinder $\rho = 1/\Omega$ (here $\Omega = 1$).

Figure 3. The roots of equation (51) are plotted in the $\alpha - \omega$ plane (curves a, b and c) along with the curve $\omega = 1/\alpha$ (curve d). The shaded regions correspond to parameter values where f has two positive roots; in the upper region between c and d these roots lie inside the inner Cauchy horizon of the black hole and in the lower region between a and b they lie outside the event horizon of the black hole.

Figure 4. The functions $\rho_{max}(\alpha)$ (curve a) and $\rho_{min}(\alpha)$ (curve b) are plotted for $0 < \alpha < 1$.

Figure 5. This and next figures illustrate qualitatively different types of motion of rigidly rotating strings in the Kerr spacetime. In all figures the inner solid circular line is the event horizon ρ_+. The nearest to the horizon dashed circle is $\rho = \rho_1$, and the outer dashed circle (if shown) is $\rho = \rho_2$. String configurations at the given moment of time are shown by solid lines with the indication of the corresponding value λ of the angular momentum parameter. The present figure illustrates Case 1(a) and shows a typical pair of string configurations ($\lambda = 0.25$, $\lambda = 0.45$) in the region $\rho_1 < \rho < \rho_2$ with no turning points. The strings have end-points at $\rho = \rho_1$ and $\rho = \rho_2$ ($\alpha = 0.5$, $\omega = 0.05$).

Figure 6. Case 1(b): A string configuration ($t = \text{const}$ slice) in the region $\rho < \rho_0$. The string has a turning point at $\rho = \rho_0$ ($\alpha = 0.5$, $\omega = 0.06$).

Figure 7. Case 2(a): String configurations inside the region $\rho_0 \leq \rho < \rho_2$ ($\lambda = 0.5$, $\lambda = 2.0$) with turning points at ρ_0. Case 3(b): A typical string configuration in the region $\rho_2 < \rho \leq \rho_0$ ($\lambda = 4.0$) with a turning point at ρ_0 ($\alpha = 0.5$, $\omega = 0.05$).

Figure 8. Case 2(b): A typical string configuration where $\rho < \rho_1$. The string is seen to spiral into the horizon $\rho = \rho_+$ from $\rho = \rho_1$ ($\alpha = 0.43$, $\omega = 0.04$).

Figure 9. A string configuration where $\rho_0 = \rho_1$. The string is seen to pass through ρ_1 and to spiral into the horizon $\rho = \rho_+$ ($\alpha = 0.43$, $\omega = 0.04$).

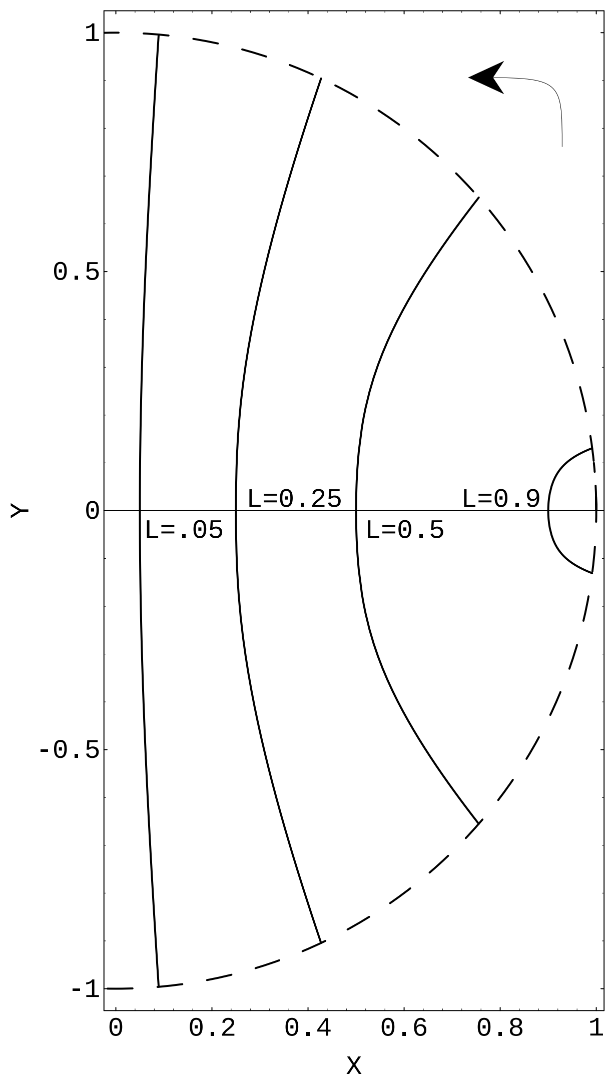


Figure 1

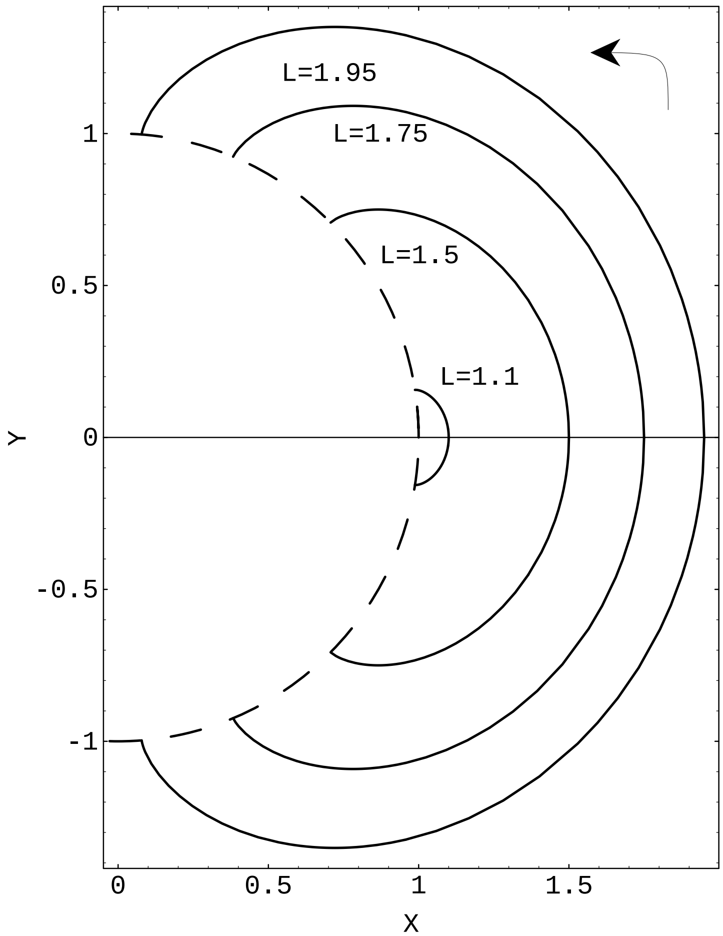
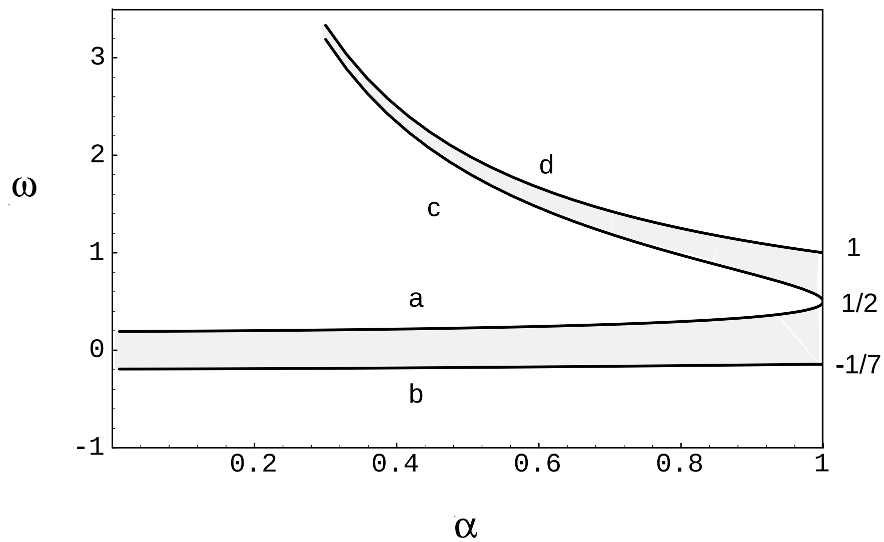


Figure 2



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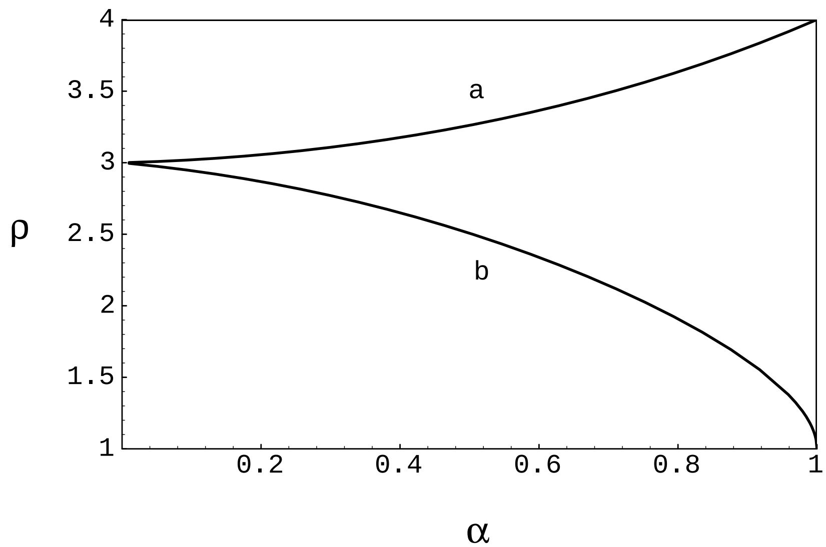

Figure 4

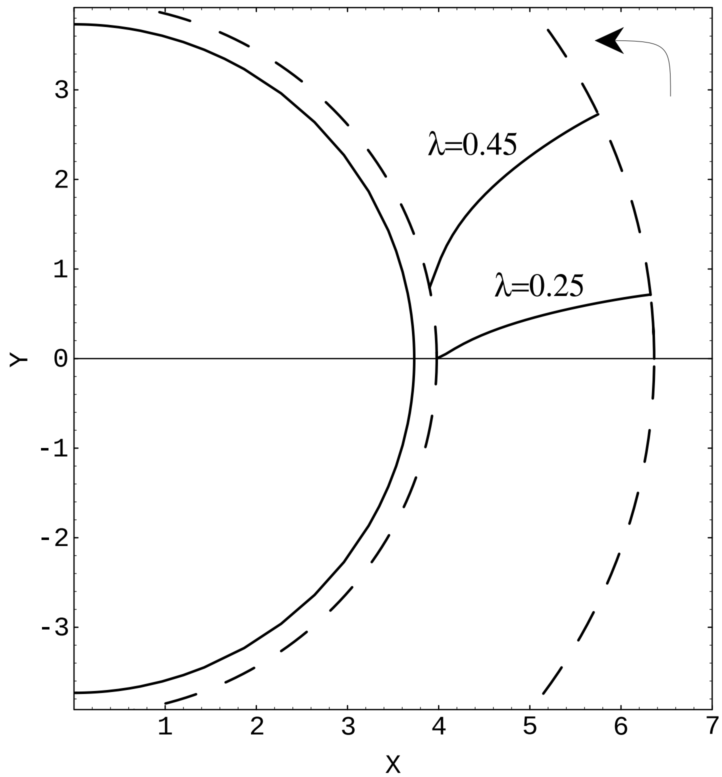

Figure 5

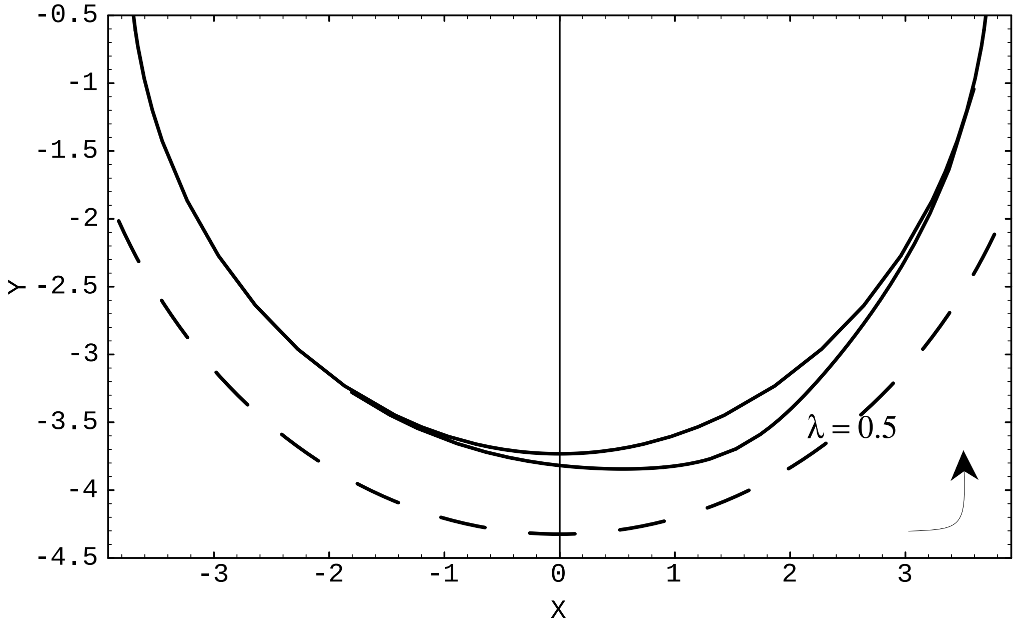

Figure 6

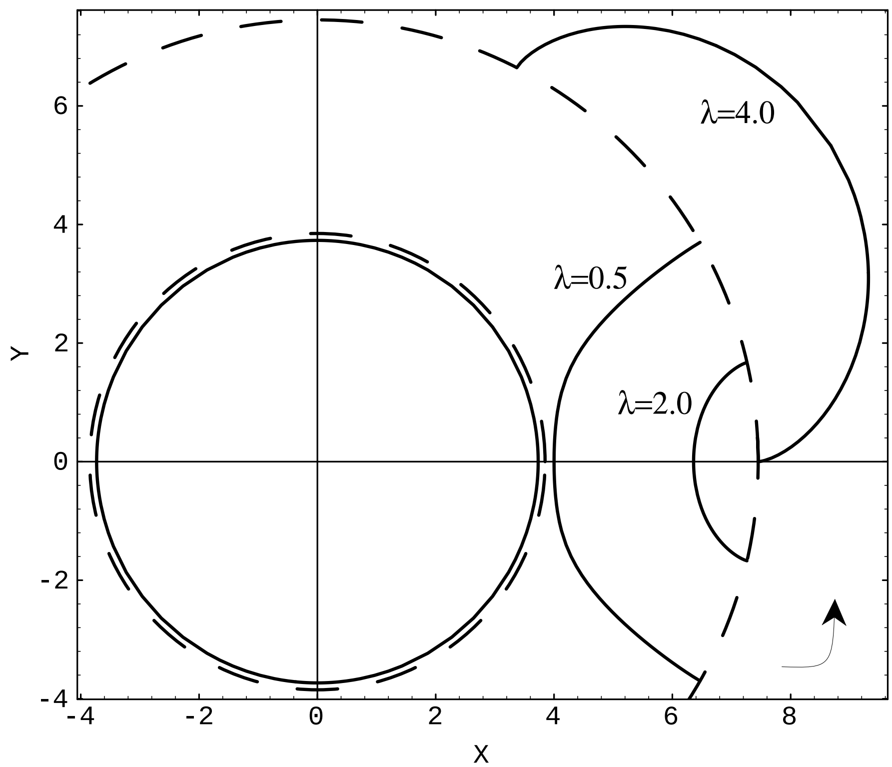

Figure 7

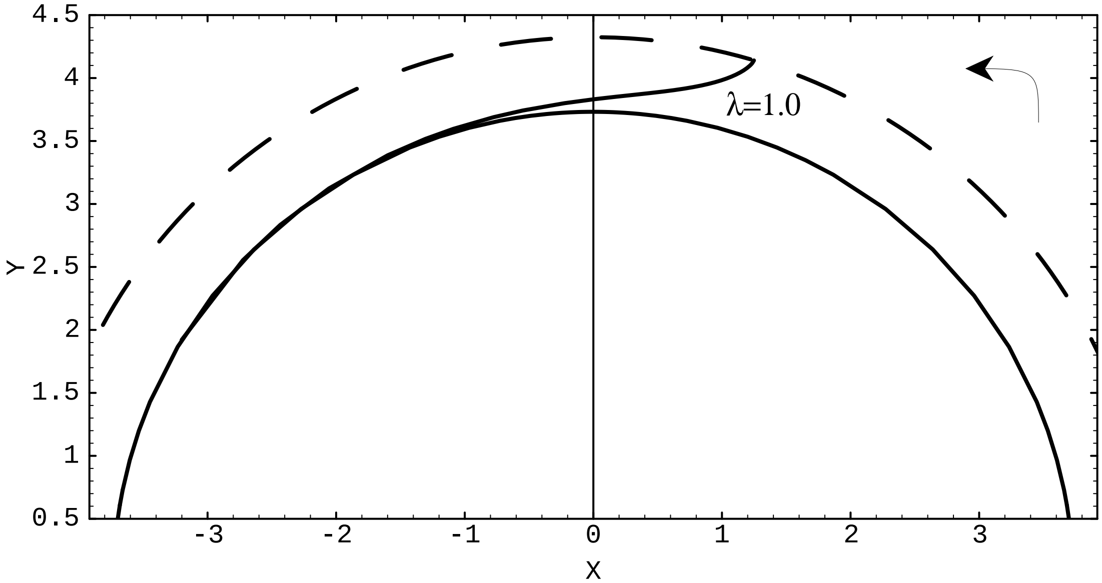

Figure 8

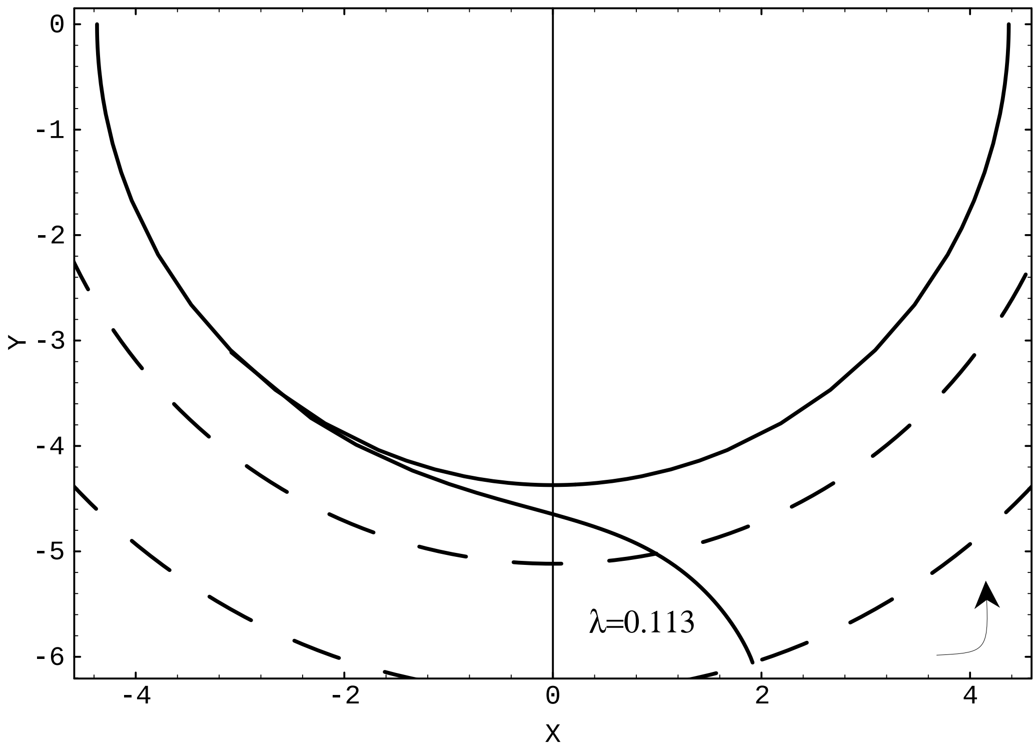

Figure 9